

Chapter I. THE PARADIGMATIC EXAMPLE:

$$\mathbf{Int}(\mathbb{Z}) = \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

The Golden Ratio and the I.V.P.s

What follows comes from a question raised by Vesslin Dimitrov [1] on *mathoverflow* in 2013:

“Are there infinitely many integer-valued polynomials dominated by 1.9^n on all $\mathbb{N}$?”

The origin of this question seems to be the fact that, for every k , the binomial coefficient $\binom{n}{k}$ satisfies $\binom{n}{k} < 2^n$. On this line, in *mathoverflow*, there are three interesting results and an open question.

Proposition 1. (David Speyer) *Whatever A , there is only a finite number of integer polynomials f such that $f(n) < A^n$ for all $n \in \mathbb{N}$.*

Proposition 2. (David Speyer) *Let $\varphi = \frac{1+\sqrt{5}}{2}$. Whatever $\epsilon > 0$, there is only a finite number of integer-valued polynomials f such that $f(n) < (\varphi - \epsilon)^n$.*

Proposition 3. (N. Elkies and D. Speyer) *Let $\varphi = \frac{1+\sqrt{5}}{2}$. Whatever $\epsilon > 0$, there are infinitely many integer-valued polynomials f such that $f(n) < (\varphi + \epsilon)^n$.*

Open Question. (W. Sawin) Are there finitely many, or not, integer-valued polynomials f such that $f(n) \leq \varphi^n$.

To go further, see [2].

References

- [1] V. Dimitrov, Are there infinitely many integer-valued polynomials dominated by 1.9^n on all $\mathbb{N}$?, *mathoverflow*, 2013.
- [2] A. Kiro and A. Nishry, Integer-valued polynomials satisfying growth constraints, *arXiv:2508.1733v1*, 24 Aug. 2025.